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# SPECTRUM OF SIZES FOR PERFECT DELETION-CORRECTING CODES*

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**Abstract.** One peculiarity with deletion-correcting codes is that perfect $t$-deletion-correcting codes of the same length over the same alphabet can have different numbers of codewords, because the balls of radius $t$ with respect to the Levenshteĭn distance may be of different sizes. There is interest, therefore, in determining all possible sizes of a perfect $t$-deletion-correcting code, given the length $n$ and the alphabet size $q$. In this paper, we determine completely the spectrum of possible sizes for perfect $q$-ary 1-deletion-correcting codes of length three for all $q$, and perfect $q$-ary 2-deletion-correcting codes of length four for almost all $q$, leaving only a small finite number of cases in doubt.

**Key words.** deletion-correcting codes, directed packings, group divisible designs, optimal codes, perfect codes

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## 1. Introduction.

When communication takes place over an insertion-deletion channel, data is lost and gained between the source and the receiver at unknown locations in the data stream. Insertion-deletion channels are rather commonplace as follows:

- *Magnetic and optical recording* [5, 37]: Random fluctuations in the motion of the recording medium, during both the read and write processes, cause timing uncertainty in the read-back signal. The resulting received data stream can miss data at certain locations.
- *Packet-switched communication* [40]: Effects of congestion control protocols and randomness of packet arrivals give rise to random packet loss.
- *DNA replication* [30, 33]: DNA can undergo deletion and insertion mutations, where bases are deleted or added, because of strand slippage and homologous recombination.
- *Music performance* [22, 32]: In music performance, pitch errors can occur. For instance, notes can be played that are not in the score (insertion errors) and notes that are specified in the score are omitted (deletion errors).

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It is therefore not surprising that the study of codes for combating deletions has continued to this day [13, 14, 15, 25, 31, 34, 42, 44, 47, 48, 49, 54, 55, 56] since its systematic treatment by Levenshteĭn [26, 27] and others [9, 10, 41, 50, 51, 52] in the 1960s.

Let $X$ be a finite set of $q$ elements and $n$ be a positive integer. A set $\mathcal{C} \subseteq X^n$ is called a *$q$-ary code of length $n$*. The set $X$ is called the *alphabet* of the code $\mathcal{C}$ and the elements of $\mathcal{C}$ are called *codewords*. The *size* of $\mathcal{C}$ is $|\mathcal{C}|$, the number of codewords it contains. For $x \in X^n$ and $0 \leq t \leq n$, let $D_t(x)$ denote the set of *$t$-th order descendants*, that is, the set of $y \in X^{n-t}$ that are obtained if any $t$ components are deleted from $x$. The $q$-ary code $\mathcal{C} \subseteq X^n$ is said to be *$t$-deletion-correcting* if $D_t(x) \cap D_t(y) = \varnothing$ for all distinct $x, y \in \mathcal{C}$. We call such a code an *$(n,t)_q$-deletion-correcting code* and denote it by $(n,t)_q$-DCC. An alternative characterization of deletion-correcting codes is via a metric known as the Levenshteĭn distance. The *Levenshteĭn distance* between two vectors $x, y \in X^n$, denoted $d_L(x,y)$, is defined as the smallest number of deletions and insertions needed to change $x$ to $y$. A code $\mathcal{C}$ is $t$-deletion-correcting if and only if $d_L(x,y) \geq 2t+1$ for all distinct $x, y \in \mathcal{C}$. Levenshteĭn [26] showed that a code capable of correcting $t$ deletions is also capable of correcting any combination of up to $t$ deletions and insertions. For this reason, deletion-correcting codes are also often called insertion/deletion-correcting codes.

Motivated by the definition of a perfect error-correcting code, Levenshteĭn [28] defines an $(n,t)_q$-DCC $\mathcal{C} \subseteq X^n$ to be *perfect* if the balls $D_t(x)$, $x \in \mathcal{C}$, partition $X^{n-t}$. However, as observed by Sloane [44], unlike in the case of error-correcting codes, perfect $t$-deletion-correcting codes of the same length over the same alphabet can have different numbers of codewords—an initially surprising fact that is explained by the balls $D_t(x)$ having different sizes for different codewords $x$. It is interesting, therefore, to ask for the possible sizes that a perfect $(n,t)_q$-DCC can have.

**2. State of affairs.** Define

$$A_q(n,t) = \max\{|\mathcal{C}| : \mathcal{C} \text{ is an } (n,t)_q\text{-DCC}\}.$$

If $\mathcal{C}$ is an $(n,t)_q$-DCC such that $|\mathcal{C}| = A_q(n,t)$, we say that $\mathcal{C}$ is *optimal*.

It is easy to see that for all $q \geq 2$ and $n \geq 1$, we have

$$A_q(n,t) = \begin{cases} 1 & \text{if } t = n, \\ q & \text{if } t = n-1. \end{cases}$$

If we define the *spectrum of sizes of a perfect* $(n,t)_q$-DCC as

$$\text{Spec}(q,n,t) = \{|\mathcal{C}| : \mathcal{C} \text{ is a perfect } (n,t)_q\text{-DCC}\},$$

then it is also easy to show that

$$\text{Spec}(q,n,t) = \begin{cases} \{1\} & \text{if } t = n, \\ \{\lceil q/n \rceil, \ldots, q\} & \text{if } t = n-1. \end{cases}$$

Hence, the problem of determining $\text{Spec}(q,n,t)$ for $t \in \{n-1, n\}$ is well solved. We consider the case $t = n-2$ in this paper. This case is nontrivial and has been treated by various researchers [5, 6, 25, 28, 29, 42, 54, 56]. The following result on the size of an optimal $(n, n-2)_q$-DCC was obtained by Bours [6].

THEOREM 2.1 (Bours [6]). *Define*

$$DU(q,n) := \left\lfloor \frac{q}{n} \left\lfloor \frac{2(q-1)}{n-1} \right\rfloor \right\rfloor + q.$$

*Then* $A_q(n, n-2) \leq DU(q,n)$ *for all* $q \geq 2$ *and* $n \geq 2$.

The existence of perfect $(3,1)_q$-DCCs has been settled by Levenshteĭn [28], where all the codes are also optimal. Bours [6] also proved the existence of two more classes of perfect $(n, n-2)_q$-DCCs, but their codes are not always optimal.

THEOREM 2.2 (Bours [6]). *The following codes exist.*

1. *A perfect* $(4,2)_q$*-DCC of size* $DU(q,4) - \Delta(q,4)$*, where*

$$\Delta(q,4) = \begin{cases} \lfloor q/4 \rfloor & \text{if } q \equiv 0 \pmod 3, \\ 0 & \text{if } q \equiv 1 \pmod 3, \\ \lfloor (q-2)/6 \rfloor & \text{if } q \equiv 2 \pmod 3 \end{cases}$$

*for all* $q$.

2. *A perfect* $(5,3)_q$*-DCC of size* $DU(q,5) - \Delta(q,5)$*, where*

$$\Delta(q,5) = \begin{cases} q/5 & \text{if } q \equiv 0 \pmod{10}, \\ 0 & \text{if } q \equiv 1 \pmod{10}, \\ (3q-16)/10 & \text{if } q \equiv 2 \pmod{10}, \\ (8q-14)/10 & \text{if } q \equiv 3 \pmod{10}, \\ (4q-6)/10 & \text{if } q \equiv 4 \pmod{10}, \\ 0 & \text{if } q \equiv 5 \pmod{10}, \\ (q-16)/10 & \text{if } q \equiv 6 \pmod{10}, \\ (6q-14)/10 & \text{if } q \equiv 9 \pmod{10} \end{cases}$$

*for all* $q \equiv 0, 1, 2, 3, 4, 5, 6,$ *or* $9 \pmod{10}$*, except possibly for* $q \in \{13, 14, 15, 16\}$.

Recently, Wang [53] proved the existence of two more classes of perfect $(n, n-2)_q$-DCCs achieving optimality.

THEOREM 2.3 (Wang [53]). *The following codes exist.*

1. *A perfect* $(4,2)_q$*-DCC of size* $DU(q,4) - \Delta(q,4)$*, where*

$$\Delta(q,4) = \begin{cases} 1 & \text{if } q = 9, \\ 0 & \text{otherwise} \end{cases}$$

*for all* $q$.

2. *A perfect* $(5,3)_q$*-DCC of size* $DU(q,5) - \Delta(q,5)$*, where*

$$\Delta(q,5) = \begin{cases} 1 & \text{if } q \equiv 7, 9 \pmod{10}, \\ 0 & \text{otherwise} \end{cases}$$

*for all* $q$*, except possibly for* $q \in \{13, 15, 19, 27, 34\}$.

Our interest in this paper is on the general problem of determining the spectrum of possible sizes of a perfect $(n, n-2)_q$-DCC. The main contributions of this paper are the determination of

1. the spectrum of sizes of a perfect $(3,1)_q$-DCC for all $q$; and
2. the spectrum of sizes of a perfect $(4,2)_q$-DCC for all but 19 values of $q$.

Our approach is design-theoretic and we begin by reviewing some notation and terminology as well as required concepts and prior results in combinatorial design theory.

## 3. Mathematical preliminaries.

**3.1. Notation and terminology.** The ring of integers modulo $n$, $\mathbb{Z}/n\mathbb{Z}$, is denoted $\mathbb{Z}_n$.

For integers $m \le n$, the set $\{m, m+1, \ldots, n\}$ is denoted $[m, n]$, and we further abbreviate $[1, n]$ to $[n]$. Given a finite set $X$ and an integer $k \in [|X|]$, we define the following:

$$\binom{X}{k} = \{A \subseteq X : |A| = k\},$$
$$X^n = \{(x_1, \ldots, x_n) : x_i \in X \text{ for all } i \in [n]\},$$
$$X_*^n = \{(x_1, \ldots, x_n) \in X^n : x_i \ne x_j,\ 1 \le i < j \le n\}.$$

Let $m \in [n]$. A vector $(x_1, \ldots, x_n) \in X^n$ is said to *contain* a vector $(y_1, \ldots, y_m) \in X^m$ if $y_i = x_{s_i}$ for some strictly increasing sequence of integers $(s_i)_{i \in [m]}$.

Given two sets $A, B \subseteq \mathbb{Z}$, $A + B$ denotes their *Minkowski sum* $\{a + b : a \in A, b \in B\}$, and for sets $A_i \subseteq \mathbb{Z}$, we denote $A_1 + \cdots + A_k$ by $\sum_{i=1}^{k} A_i$.

For $A = \{a_1, \ldots, a_k\} \subseteq \mathbb{Z}$, $A_\uparrow$ denotes the vector whose components are the elements of $A$ sorted in ascending order, that is, $A_\uparrow = (a_1, \ldots, a_k)$, where $a_1 < \cdots < a_k$. If $A_\uparrow = (a_1, \ldots, a_k)$, then the vector $(a_k, \ldots, a_1)$ is denoted $A_\downarrow$.

**3.2. Set systems.** A *set system* is a pair $(X, \mathcal{A})$, where $X$ is a finite set of *points* and $\mathcal{A} \subseteq 2^X$, whose elements are called *blocks*. The *order* of the set system is $|X|$, the number of points. For a set of nonnegative integers $K$, a set system $(X, \mathcal{A})$ is said to be $K$*-uniform* if $|A| \in K$ for all $A \in \mathcal{A}$.

Let $(X, \mathcal{A})$ be a set system, and let $\mathcal{B} \subseteq \mathcal{A}$. If $\sum_{B \in \mathcal{B}} |B| = |X|$ and $\cup_{B \in \mathcal{B}} B = X$, then $\mathcal{B}$ is called a *parallel class* of $(X, \mathcal{A})$. If $\sum_{B \in \mathcal{B}} |B| = |X| - 1$ and $\cup_{B \in \mathcal{B}} B = X \setminus \{x\}$ for some $x \in X$, then $\mathcal{B}$ is called a *near-parallel class* of $(X, \mathcal{A})$. If $\mathcal{A}$ can be partitioned into parallel classes, then $(X, \mathcal{A})$ is said to be *resolvable*. If $\mathcal{A}$ can be partitioned into near-parallel classes, then $(X, \mathcal{A})$ is said to be *near-resolvable*.

**3.3. Packings.** A *packing*, $\mathrm{P}(q, n, \lambda)$, of order $q$, block size $n$, and index $\lambda$ is a set system $(X, \mathcal{A})$ of order $q$ such that the following conditions are satisfied:

(a) $\mathcal{A}$ is a subset of elements of $\binom{X}{n}$, called *blocks*; and

(b) every element of $\binom{X}{2}$ is contained in at most $\lambda$ blocks.

If we replace $\binom{X}{n}$ by $X^n$ in condition (a) and replace $\binom{X}{2}$ by $X^2$ in condition (b), we obtain the definition of a *directed packing of order $q$, block size $n$, and index* $\lambda$, denoted $\mathrm{DP}(q, n, \lambda)$. If we replace $\binom{X}{n}$ by $X_*^n$ in condition (a) and replace $\binom{X}{2}$ by $X_*^2$ in condition (b), we have the definition of a *restricted directed packing of order $q$, block size $n$, and index* $\lambda$, denoted $\mathrm{RDP}(q, n, \lambda)$. Throughout this paper, we adopt the convention that $\lambda$ is omitted from our notation if $\lambda = 1$. A *perfect* $\mathrm{P}(q, n, \lambda)$ is one for which

(b′) every element of $\binom{X}{2}$ is contained in exactly $\lambda$ blocks.

*Perfect* $\mathrm{DP}(q, n, \lambda)$ and *perfect* $\mathrm{RDP}(q, n, \lambda)$ are similarly defined (by replacing $\binom{X}{2}$ in condition (b′) with $X^2$ and $X_*^2$, respectively).

A perfect $\mathrm{P}(q, n, \lambda)$ is known as a *balanced incomplete block design* (BIBD) and is denoted $\mathrm{B}(q, n, \lambda)$. A perfect $\mathrm{RDP}(q, n, \lambda)$ is known as a *directed balanced incomplete block design* (DBIBD) and is denoted $\mathrm{DB}(q, n, \lambda)$.

THEOREM 3.1 (Hanani [19]). *Let $n \in \{4, 5\}$ and $q \ge n$. Then there exists a* $\mathrm{B}(q, n, \lambda)$ *if and only if* $\lambda q(q-1) \equiv 0 \pmod{n(n-1)}$ *and* $\lambda(q-1) \equiv 0 \pmod{n-1}$, *except for* $(q, n, \lambda) = (15, 5, 2)$.

The maximum number of blocks in a $\mathrm{P}(q,n,\lambda)$, $\mathrm{DP}(q,n,\lambda)$, and $\mathrm{RDP}(q,n,\lambda)$ is denoted by $D(q,n,\lambda)$, $DD(q,n,\lambda)$, and $RDD(q,n,\lambda)$, respectively. A $\mathrm{P}(q,n,\lambda)$ with $D(q,n,\lambda)$ blocks, a $\mathrm{DP}(q,n,\lambda)$ with $DD(q,n,\lambda)$ blocks, and an $\mathrm{RDP}(q,n,\lambda)$ with $RDD(q,n,\lambda)$ blocks are called *optimal.* Johnson [24] and Schőnheim [39] showed that

$$D(q,n,\lambda) \leq \left\lfloor \frac{q}{n} \left\lfloor \frac{\lambda(q-1)}{n-1} \right\rfloor \right\rfloor =: U(q,n,\lambda).$$

Using similar counting arguments, the following Johnson-type bound holds:

$$RDD(q,n,\lambda) \leq \left\lfloor \frac{q}{n} \left\lfloor \frac{2\lambda(q-1)}{n-1} \right\rfloor \right\rfloor =: RDU(q,n,\lambda). \tag{3.1}$$

THEOREM 3.2 (Skillicorn [43]; Assaf, Shalaby, and Yin [3]). *For all positive integers $q$, $RDD(q,4) = RDU(q,4)$ except $RDD(9,4) = RDU(9,4) - 1$.*

Directed packings have featured strongly in the study of $(n,n-2)_q$-DCCs due to the following equivalence, for which we give a proof for the sake of completeness.

PROPOSITION 3.3 (folklore). *$\mathcal{C} \subseteq X^n$ is an $(n,n-2)_q$-DCC if and only if $(X,\mathcal{C})$ is a $\mathrm{DP}(q,n)$.*

*Proof.* We need only check that every element of $X^2$ is contained in at most one block in $\mathcal{C}$. Suppose this is not the case. Then there are two blocks $\mathsf{u},\mathsf{v} \in \mathcal{C}$ containing a common vector $(a,b) \in X^2$. The Levenshteĭn distance between $\mathsf{u}$ and $\mathsf{v}$ is therefore at most $2(n-2)$ since we can obtain $\mathsf{v}$ from $\mathsf{u}$ by deleting $n-2$ components from $\mathsf{u}$ leaving $(a,b)$ and then inserting $n-2$ components to get $\mathsf{v}$. Hence $\mathcal{C}$ is not $(n-2)$-deletion-correcting since any $(n-2)$-deletion-correcting code must have a Levenshteĭn distance of at least $2(n-2)+1$ between every pair of codewords. ∎

In view of Proposition 3.3, $\mathcal{C}$ can interchangeably be called an $(n,n-2)_q$-DCC or a $\mathrm{DP}(q,n)$. The following is also now evident.

PROPOSITION 3.4. *$\mathcal{C} \subseteq X^n$ is a perfect $(n,n-2)_q$-DCC if and only if $(X,\mathcal{C})$ is a perfect $\mathrm{DP}(q,n)$.*

*Proof.* If $\mathcal{C}$ is a perfect $(n,n-2)_q$-DCC, then the balls $D_{n-2}(\mathsf{u})$, $\mathsf{u} \in \mathcal{C}$, partition $X^2$, which is just another way of saying that every element of $X^2$ is contained in exactly one block. ∎

Restricted directed packings do not directly give useful deletion-correcting codes for real-world communication channels since the constraint of not allowing repeated components in codewords is seldom imposed by such channels. For example, in a noisy binary channel, it is necessary for codewords to contain repeated components for any meaningful communication to take place. However, restricted directed packings are important because they can be used as subcodes of deletion-correcting codes, allowing repeated components, and, moreover, an optimal restricted directed packing can be used to construct an optimal directed packing.

LEMMA 3.5. *There exists an $\mathrm{RDP}(q,n)$ having $M$ blocks if and only if there exists a $\mathrm{DP}(q,n)$ having $M+q$ blocks. Furthermore, if the $\mathrm{RDP}(q,n)$ is a $\mathrm{DB}(q,n)$, then the $\mathrm{DP}(q,n)$ is perfect and optimal.*

*Proof.* If $(X,\mathcal{A})$ is an $\mathrm{RDP}(q,n)$ with $M$ blocks, then $(X,\mathcal{A}\cup\{(x,\ldots,x) \in X^n\})$ is a $\mathrm{DP}(q,n)$ with $M+q$ blocks. It is easy to check that this $\mathrm{DP}(q,n)$ is perfect if $(X,\mathcal{A})$ is a $\mathrm{DB}(q,n)$. To show that this perfect $\mathrm{DP}(q,n)$ is optimal, note that $M = RDU(q,n)$ if $(X,\mathcal{A})$ is a $\mathrm{DB}(q,n)$.

Conversely, if $(X,\mathcal{A})$ is a $\mathrm{DP}(q,n)$ with $M+q$ blocks, then deleting all the blocks containing the pairs $(x,x) \in X^2$ gives an $\mathrm{RDP}(q,n)$ with at least $M$ blocks. ∎

The following are two consequences of the above lemma.

COROLLARY 3.6. *$DD(q,n) = RDD(q,n) + q$.*

COROLLARY 3.7. *For $n \in \{3,4,5,6\}$, there exists a* DP$(q,n)$ *that is perfect and optimal whenever $2q(q-1) \equiv 0 \pmod{n(n-1)}$ and $2(q-1) \equiv 0 \pmod{n-1}$, except for $(q,n) \in \{(15,5),(21,6)\}$.*

*Proof.* The proof follows from the existence of DB$(q,n)$ for $n \in \{3,4,5,6\}$ [4, 23, 45, 46]. ∎

The *leave graph* of a DP$(q,n,\lambda)$ $(X,\mathcal{A})$ is a directed loopless multigraph $\Gamma$ with $V(\Gamma) = X$, and an edge $(a,b) \in X_*^2$ appears $\lambda - s$ times in $E(\Gamma)$ if and only if $(a,b)$ appears in $s$ blocks of $\mathcal{A}$.

### 3.4. Pairwise balanced designs and group divisible designs.

A *pairwise balanced design* (PBD) with set of block sizes $K$ is a $K$-uniform set system $(X,\mathcal{A})$, such that every element of $\binom{X}{2}$ is contained in exactly one block of $\mathcal{A}$. A PBD of order $q$ and set of block sizes $K$ is denoted PBD$(q,K)$. If an element $k \in K$ is "starred" (written $k^\star$), it means that the PBD has exactly one block of size $k$.

THEOREM 3.8 (Brouwer [7]). *A* PBD$(q,\{4,7^\star\})$ *exists if and only if $q \equiv 7$ or $10 \pmod{12}$, and $q \geq 22$.*

Let $\lambda$ be a positive integer and $K \subseteq \mathbb{Z}$. A *group divisible design* (GDD) *with set of block sizes $K$ and index $\lambda$*, denoted $(K,\lambda)$-GDD, is a triple $(X,\mathcal{G},\mathcal{A})$, where

(c) $\mathcal{A}$ is a set of elements from $\cup_{k\in K}\binom{X}{k}$, called *blocks*;
(d) $\mathcal{G} = \{G_1,\dots,G_s\}$ is a partition of $X$ into subsets, called *groups*;
(e) every element of $\binom{X}{2}$ not contained in a group is contained in exactly $\lambda$ blocks, and
(f) no block contains more than one point from any group.

The *type* of a GDD $(X,\mathcal{G},\mathcal{A})$ is the multiset $\{|G| : G \in \mathcal{G}\}$. When it is more convenient, we use the exponential notation to describe the type of a GDD: a GDD of type $g_1^{t_1}\cdots g_s^{t_s}$ is a GDD where there are exactly $t_i$ groups of cardinality $g_i$, $i \in [s]$.

If we replace $\cup_{k\in K}\binom{X}{k}$ by $\cup_{k\in K}X_*^k$ in condition (c) and replace $\binom{X}{2}$ by $X_*^2$ in condition (e), we get the definition of a *directed group divisible design* (DGDD) *with set of block sizes $K$ and index $\lambda$*, which we denote by $(K,\lambda)$-DGDD.

Again, we omit $\lambda$ from our notation if $\lambda = 1$. Hence, we write $K$-GDD and $K$-DGDD instead of $(K,1)$-GDD and $(K,1)$-DGDD.

A $\{k\}$-GDD of type $n^k$ is called a *transversal design* and is denoted TD$(k,n)$.

A $(K,\lambda)$-GDD $(X,\mathcal{G},\mathcal{A})$ is said to be *resolvable* if $(X,\mathcal{A})$ is resolvable.

We have the following simple observation.

LEMMA 3.9. *If there exists a $(K,\lambda)$-GDD of type $g_1^{t_1}\cdots g_s^{t_s}$, then there also exists a $(K,\lambda)$-DGDD of type $g_1^{t_1}\cdots g_s^{t_s}$.*

*Proof.* If $(X,\mathcal{G},\mathcal{A})$ is a $(K,\lambda)$-GDD of type $g_1^{t_1}\cdots g_s^{t_s}$, then $(X,\mathcal{G},\mathcal{B})$, where $\mathcal{B} = \{A_\uparrow, A_\downarrow : A \in \mathcal{A}\}$ is a $(K,\lambda)$-DGDD of type $g_1^{t_1}\cdots g_s^{t_s}$. ∎

Another useful construction for DGDDs from GDDs is the following "averaging construction."

LEMMA 3.10 (averaging construction). *Let $x \equiv y \pmod 2$, $x \geq y$. If there exist $K$-GDDs of types $g^n x^1$ and $g^n y^1$, then there exists a $K$-DGDD of type $g^n(\frac{x+y}{2})^1$.*

*Proof.* Let $V = [gn]$, and let $\{G_1,\dots,G_n\}$ be a partition of $V$ into $n$ parts of size $g$. Let $X = \{a_1,\dots,a_x\}$ be disjoint from $V$, and let $Y = \{a_1,\dots,a_y\}$, $Z = \{a_1,\dots,a_{(x+y)/2}\}$. Suppose that $(V\cup X, \{G_1,\dots,G_n,X\},\mathcal{A})$ is a $K$-GDD of type $g^n x^1$ and $(V\cup Y,\{G_1,\dots,G_n,Y\},\mathcal{B})$ is a $K$-GDD of type $g^n y^1$.

For each block $A \in \mathcal{A}$ such that $a_i \in A$ and $i \in [(x+y)/2]$, let $T_1(A)$ be the ordered tuple $(a_i,u_1,\dots,u_k)$ such that $\{a_i,u_1,\dots,u_k\} = A$ and $u_1 < \cdots < u_k$. For each block $A \in \mathcal{A}$ such that $a_i \in A$ and $i \in [(x+y)/2+1, x]$, let $T_2(A)$ be the ordered tuple $(u_1, \ldots, u_k, a_i)$ such that $\{a_i, u_1, \ldots, u_k\} = A$ and $u_1 < \cdots < u_k$. For each block $A \in \mathcal{A}$ such that $a_i \notin A$ for all $i \in [x]$, let $T_3(A) = A_{\uparrow}$.

For each block $B \in \mathcal{B}$ such that $a_i \in B$ and $i \in [y]$, let $T_4(B)$ be the ordered tuple $(u_1, \ldots, u_k, a_i)$ such that $\{a_i, u_1, \ldots, u_k\} = B$ and $u_1 > \cdots > u_k$. For each block $B \in \mathcal{B}$ such that $a_i \notin B$ for all $i \in [y]$, let $T_5(B) = B_{\downarrow}$.

Define

$$\mathcal{C} = \{T_1(A), T_2(A), T_3(A) : A \in \mathcal{A}\} \cup \{T_4(B), T_5(B) : B \in \mathcal{B}\}.$$

Now, for each block in $\mathcal{C}$ that contains $a_i$, $i \in [x]$, relabel the point $a_{x+1-j}$ as $a_{y+j}$, $j \in [(x-y)/2]$, and denote the resulting blocks as $\mathcal{D}$. Then $(V \cup Z, \{G_1, \ldots, G_n, Z\}, \mathcal{D})$ is a $K$-DGDD of type $g^n(\frac{x+y}{2})^1$. ◻

The following results on the existence of TDs, GDDs, and DGDDs are needed.

THEOREM 3.11 (folklore; see [1]). *There exists a* TD$(k, n)$ *for all* $k \leq n + 1$ *whenever* $n$ *is a prime power.*

THEOREM 3.12 (Zhu [57]). *The necessary and sufficient conditions for the existence of a* $(\{3\}, \lambda)$*-GDD of type* $m^u$ *are*

1. $u \geq 3$;
2. $\lambda(u-1)m \equiv 0 \pmod 2$; *and*
3. $\lambda u(u-1)m^2 \equiv 0 \pmod 6$.

THEOREM 3.13 (Assaf and Hartman [2]; Rees and Stinson [35]; Rees [36]). *A resolvable* $(\{3\}, \lambda)$*-GDD of type* $g^{v/g}$ *exists if and only if* $\lambda(v-g) \equiv 0 \pmod 2$, $v \equiv 0 \pmod 3$, *and* $(\lambda, g, v) \notin \{(1, 2, 12), (1, 6, 18)\} \cup \{(2j+1, 2, 6), (4j+2, 1, 6) : j \geq 0\}$.

THEOREM 3.14 (Colbourn, Hoffman, and Rees [11]). *Let* $g$, $t$, *and* $u$ *be nonnegative integers. There exists a* $\{3\}$*-GDD of type* $g^t u^1$ *if and only if all of the following conditions are satisfied:*

1. *if* $g > 0$, *then* $t \geq 3$, *or* $t = 2$ *and* $u = g$, *or* $t = 1$ *and* $u = 0$, *or* $t = 0$;
2. $u \leq g(t-1)$ *or* $gt = 0$;
3. $g(t-1) + u \equiv 0 \pmod 2$ *or* $gt = 0$;
4. $gt \equiv 0 \pmod 2$ *or* $u = 0$;
5. $g^2\binom{t}{2} + gtu \equiv 0 \pmod 3$.

THEOREM 3.15 (Brouwer, Schrijver, and Hanani [8]). *There exists a* $\{4\}$*-GDD of type* $g^t$ *if and only if* $t \geq 4$ *and*

1. $g \equiv 1$ *or* $5 \pmod 6$ *and* $t \equiv 1$ *or* $4 \pmod{12}$; *or*
2. $g \equiv 2$ *or* $4 \pmod 6$ *and* $t \equiv 1 \pmod 3$; *or*
3. $g \equiv 3 \pmod 6$ *and* $t \equiv 0$ *or* $1 \pmod 4$; *or*
4. $g \equiv 0 \pmod 6$,

*with the two exceptions of types* $2^4$ *and* $6^4$, *for which* $\{4\}$*-GDDs do not exist.*

THEOREM 3.16 (Brouwer [7]). *There exists a* $\{4\}$*-GDD of type* $2^t 5^1$ *if and only if* $t \equiv 0 \pmod 3$, *and* $t \geq 9$.

THEOREM 3.17 (Ge and Ling [17]). *There exists a* $\{4\}$*-GDD of type* $12^u m^1$ *for every* $u \geq 4$ *and* $m \equiv 0 \pmod 3$, $0 \leq m \leq 6(u-1)$.

THEOREM 3.18 (Ge and Rees [18]). *There exists a* $\{4\}$*-GDD of type* $6^u m^1$ *for every* $u \geq 4$ *and* $m \equiv 0 \pmod 3$, *with* $0 \leq m \leq 3u - 3$, *except for* $(u, m) = (4, 0)$ *and except possibly for* $(u, m) \in \{(7, 15), (11, 21), (11, 24), (11, 27), (13, 27), (13, 33), (17, 39), (17, 42), (19, 45), (19, 48), (19, 51), (23, 6), (23, 63)\}$.

THEOREM 3.19 (Sarvate [38]). *There exists a* $\{4\}$*-DGDD of type* $g^{v/g}$ *if and only if* $v \geq 4g$, $v \equiv 0 \pmod g$, $v \equiv g \pmod 3$, *and* $v(v-g) \equiv 0 \pmod 6$.

**3.5. Incomplete objects.** A *perfect* IDP *of order $q$ and block size $n$, with a hole of size $h$*, denoted *perfect* IDP$(q : h, n)$, is a triple $(X, Y, \mathcal{A})$, where $(X, \mathcal{A})$ is a DP$(q, n)$ and $Y$ is an $h$-subset of $X$ such that

1. no element of $Y^2$ is contained in any block of $\mathcal{A}$; and
2. every element of $X^2 \setminus Y^2$ is contained in exactly one block of $\mathcal{A}$.

Note that a perfect IDP$(q : 0, n)$ is equivalent to a perfect DP$(q, n)$.

**3.6. Directability.** Suppose one is given an RDP$(q, n, \lambda)$ $(X, \mathcal{A})$ and we replace each block $(a_1, \ldots, a_n) \in \mathcal{A}$ by the $n$-subset $\{a_1, \ldots, a_n\}$. Then the resulting collection of $n$-subsets contains every element of $\binom{X}{2}$ at most $2\lambda$ times. This produces a P$(q, n, 2\lambda)$, the *underlying packing* of the RDP$(q, n, \lambda)$. A P$(q, n, 2\lambda)$ is *directable* if it is the underlying packing of some RDP$(q, n, \lambda)$.

Harms and Colbourn [20, 21] developed an algorithm for directing B$(q, 3, 2\lambda)$ that has the following important consequence.

THEOREM 3.20 (see [12]). *Every packing* P$(q, 3, 2\lambda)$ *is directable.*

**4. Perfect incomplete directed packings of small order.** Here, we show the existence of some perfect IDPs of small orders that are required for our constructions in subsequent sections.

LEMMA 4.1. *There exists a perfect* IDP$(13 : 1, 4)$.

*Proof.* Take the point set to be $\mathbb{Z}_{12} \cup \{\infty\}$. The orbits of the nine blocks

$$\begin{array}{ccc} (0,3,6,0) & (3,1,8,3) & (6,5,6,3) \\ (9,3,9,2) & (0,9,11,4) & (6,7,10,2) \\ (9,0,8,10) & (3,\infty,5,10) & (9,7,\infty,6) \end{array}$$

under the action of the group $\langle\alpha\rangle$, where

$$\alpha = (0\ 1\ 2)(3\ 4\ 5)(6\ 7\ 8)(9\ 10\ 11)(\infty),$$

together with the two additional blocks, $(\infty, 0, 1, 2)$ and $(0, 1, 2, \infty)$, give a perfect IDP$(13 : 1, 4)$, with $\{\infty\}$ as the hole of size one. ∎

LEMMA 4.2. *There exists a perfect* IDP$(28 : 4, 4)$.

*Proof.* Take the point set to be $(\mathbb{Z}_{12} \times \{0, 1\}) \cup \{\infty_0, \infty_1, \infty_2, \infty_3\}$. The orbits of the nine blocks

$$\begin{array}{ccc} ((0,1),(3,1),(0,1),(5,0)) & ((0,0),(0,1),(1,1),(0,0)) & ((0,0),(1,0),(5,1),(11,0)) \\ ((0,1),(6,1),(5,1),(3,0)) & ((0,0),(11,1),(7,0),(9,1)) & (\infty_0,(0,0),(5,0),(8,1)) \\ (\infty_0,(0,1),(1,0),(7,1)) & ((0,0),(7,1),(2,0),\infty_0) & ((0,1),(4,0),(2,1),\infty_0) \end{array}$$

under the action of the group $\langle\alpha\rangle$, where

$$\alpha(x) = \begin{cases} (y+1, i) & \text{if } x = (y, i) \in \mathbb{Z}_{12} \times \{0, 1\}, \\ \infty_{i+1 \pmod 3} & \text{if } x = \infty_i,\ i \in \{0, 1, 2\}, \end{cases}$$

together with the 22 additional blocks

$$\begin{array}{cc} (\infty_3,(0,0),(4,0),(8,0)) & ((8,0),(4,0),(0,0),\infty_3) \\ (\infty_3,(1,0),(5,0),(9,0)) & ((9,0),(5,0),(1,0),\infty_3) \\ (\infty_3,(2,0),(6,0),(10,0)) & ((10,0),(6,0),(2,0),\infty_3) \\ (\infty_3,(3,0),(7,0),(11,0)) & ((11,0),(7,0),(3,0),\infty_3) \\ (\infty_3,(0,1),(4,1),(8,1)) & ((8,1),(4,1),(0,1),\infty_3) \\ (\infty_3,(1,1),(5,1),(9,1)) & ((9,1),(5,1),(1,1),\infty_3) \\ (\infty_3,(2,1),(6,1),(10,1)) & ((10,1),(6,1),(2,1),\infty_3) \\ (\infty_3,(3,1),(7,1),(11,1)) & ((11,1),(7,1),(3,1),\infty_3) \\ ((0,0),(3,0),(6,0),(9,0)) & ((9,0),(6,0),(3,0),(0,0)) \end{array}$$

$$((1,0),(4,0),(7,0),(10,0)) \quad ((10,0),(7,0),(4,0),(1,0))$$
$$((2,0),(5,0),(8,0),(11,0)) \quad ((11,0),(8,0),(5,0),(2,0))$$

give a perfect IDP$(28:4,4)$ with $\{\infty_0,\infty_1,\infty_2,\infty_3\}$ as the hole of size four. ∎

LEMMA 4.3. *There exists a perfect* IDP$(40:4,4)$.

*Proof.* Take the point set to be $(\mathbb{Z}_{18}\times\{0,1\})\cup\{\infty_0,\ \infty_1,\ \infty_2,\ \infty_3\}$. The orbits of the 13 blocks

$$\begin{array}{cc}
((0,1),(11,1),(9,1),(15,0)) & ((0,1),(3,1),(0,1),(5,0))\\
((0,1),(2,1),(7,1),(10,0)) & ((0,0),(2,0),(4,1),(5,0))\\
((0,0),(4,0),(3,1),(15,0)) & ((0,1),(9,0),(4,1),(17,1))\\
((0,0),(17,0),(11,1),(7,0)) & ((0,0),(0,1),(1,1),(0,0))\\
((0,0),(16,1),(14,0),(6,1)) & ((0,0),(13,0),(9,1),\infty_0)\\
((1,1),(14,0),(11,1),\infty_0) & (\infty_0,(0,0),(5,1),(16,0))\\
(\infty_0,(1,1),(8,0),(15,1)) &
\end{array}$$

under the action of the group $\langle\alpha\rangle$, where

$$\alpha(x)=\begin{cases}(y+1,i) & \text{if } x=(y,i)\in\mathbb{Z}_{18}\times\{0,1\},\\ \infty_{i+1 \pmod 3} & \text{if } x=\infty_i,\ i\in\{0,1,2\},\end{cases}$$

and the orbit of the block $((0,0),(9,0),(1,0),(10,0))$ under the action of $\langle\beta\rangle$, where

$$\beta((y,i))=(y+2,i),$$

together with the 24 blocks

$$\begin{array}{ccc}
(\infty_3,(0,0),(6,0),(12,0)) & (\infty_3,(0,1),(6,1),(12,1)) & (\infty_3,(1,0),(7,0),(13,0))\\
(\infty_3,(1,1),(7,1),(13,1)) & (\infty_3,(2,0),(8,0),(14,0)) & (\infty_3,(2,1),(8,1),(14,1))\\
(\infty_3,(3,0),(9,0),(15,0)) & (\infty_3,(3,1),(9,1),(15,1)) & (\infty_3,(4,0),(10,0),(16,0))\\
(\infty_3,(4,1),(10,1),(16,1)) & (\infty_3,(5,0),(11,0),(17,0)) & (\infty_3,(5,1),(11,1),(17,1))\\
((12,0),(6,0),(0,0),\infty_3) & ((12,1),(6,1),(0,1),\infty_3) & ((13,0),(7,0),(1,0),\infty_3)\\
((13,1),(7,1),(1,1),\infty_3) & ((14,0),(8,0),(2,0),\infty_3) & ((14,1),(8,1),(2,1),\infty_3)\\
((15,0),(9,0),(3,0),\infty_3) & ((15,1),(9,1),(3,1),\infty_3) & ((16,0),(10,0),(4,0),\infty_3)\\
((16,1),(10,1),(4,1),\infty_3) & ((17,0),(11,0),(5,0),\infty_3) & ((17,1),(11,1),(5,1),\infty_3)
\end{array}$$

give a perfect IDP$(40:4,4)$ with $\{\infty_0,\infty_1,\infty_2,\infty_3\}$ as the hole of size four. ∎

**5. Bounds on sizes of perfect $(n-2)$-deletion-correcting codes.** In the remaining sections of this paper, we determine $\mathrm{Spec}(q,n,n-2)$ for all $q$ when $n=3$, and for all $q$ with a small finite number of possible exceptions when $n=4$. We begin by establishing the extremal elements of $\mathrm{Spec}(q,n,n-2)$. Define

$$DL(q,n):=\left\lceil\frac{q}{n}\left\lceil\frac{2q}{n-1}\right\rceil\right\rceil.$$

Then the following holds.

PROPOSITION 5.1. $\mathrm{Spec}(q,n,n-2)\subseteq[DL(q,n),DU(q,n)]$ *for all* $q$.

*Proof.* The upper bound follows from Theorem 2.1. To establish the lower bound, let $\mathcal{C}\subseteq X^n$ be a perfect $(n,n-2)_q$-DCC, and let $r_x$ denote the number of codewords in $\mathcal{C}$ containing the point $x\in X$. For any $x\in X$, there is exactly one codeword $\mathsf{u}\in\mathcal{C}$ containing the pair $(x,x)$. The maximum number of distinct pairs of the form

$(x, a)$ and $(a, x)$ that $\mathsf{u}$ can contain is $2n-3$, achieved when $\mathsf{u} = (x, a_1, \ldots, a_{n-2}, x)$, where $a_1, \ldots, a_{n-2}$ are all distinct and different from $x$. The other codewords in $\mathcal{C}$ containing $x$ each contains at most $n-1$ pairs either of the form $(x, a)$ or $(a, x)$. Hence, $(n-1)(r_x - 1) + 2n - 3 \geq 2q - 1$, which implies $r_x + 1 \geq \lceil 2q/(n-1) \rceil$. Hence,

$$n|\mathcal{C}| = \sum_{x \in X} (r_x + 1) \geq \sum_{x \in X} \left\lceil \frac{2q}{n-1} \right\rceil = q \left\lceil \frac{2q}{n-1} \right\rceil,$$

giving $|\mathcal{C}| \geq DL(q, n)$. ∎

Let

$$I(q, n) = [DL(q, n), DU(q, n)].$$

We show in subsequent sections that, in fact,

$$\mathrm{Spec}(q, 3, 1) = I(q, 3) \quad \text{and} \quad \mathrm{Spec}(q, 4, 2) = I(q, 4)$$

for almost all $q$, leaving only a small finite number of cases in doubt.

The following "filling-in-groups" construction is useful.

PROPOSITION 5.2. *If an $\{n\}$-DGDD of type $\{g_1, \ldots, g_s\}$ exists, then*

(5.1)

$$\left\{ \frac{2}{\binom{n}{2}} \left( \binom{\sum_{i=1}^{s} g_i}{2} - \sum_{i=1}^{s} \binom{g_i}{2} \right) \right\} + \sum_{i=1}^{s} \mathrm{Spec}(g_i, n, n-2) \subseteq \mathrm{Spec}\left( \sum_{i=1}^{s} g_i, n, n-2 \right).$$

*Proof.* Let $(X, \mathcal{A})$ be an $\{n\}$-DGDD of type $\{g_1, \ldots, g_s\}$. We produce a perfect $\mathrm{DP}(\sum_{i=1}^{s} g_i, n)$ as follows. For each group $G_i \in \mathcal{G}$ of size $g_i$, we add the blocks of a perfect $\mathrm{DP}(g_i, n)$ $(G_i, \mathcal{B}_i)$ to $\mathcal{A}$. By using perfect $\mathrm{DP}(g_i, n)$'s of different sizes, we obtain (5.1), noting that the first term in (5.1) is the quantity $|\mathcal{A}|$. ∎

With a proof similar to that of Proposition 5.2, we have the following construction.

PROPOSITION 5.3. *Let $h$ be a positive integer. If we have*

1. *an $\{n\}$-DGDD of type $\{g_1, \ldots, g_s\}$;*
2. *a perfect* $\mathrm{IDP}(g_i + h : h, n)$ *having $t_i$ blocks for $1 \leq i \leq s-1$; and*
3. *a perfect* $\mathrm{DP}(g_s + h, n)$ *having $t$ blocks,*

*then*

$$\left\{ \frac{2}{\binom{n}{2}} \left( \binom{\sum_{i=1}^{s} g_i}{2} - \sum_{i=1}^{s} \binom{g_i}{2} \right) \right\} + t + \sum_{i=1}^{s-1} t_i \in \mathrm{Spec}\left( h + \sum_{i=1}^{s} g_i, n, n-2 \right).$$

**6. The spectrum $\mathrm{Spec}(q, 3, 1)$.** We begin by determining $I(q, 3)$ for some small $q$.

LEMMA 6.1. *For $q \in \{1, 2, 3, 4, 5, 6, 8, 11, 14\}$, we have* $\mathrm{Spec}(q, 3, 1) = I(q, 3)$.

*Proof.* For $q \in \{1, 2, 3, 4, 5, 6, 8\}$, perfect $(3, 1)_q$-DCCs having number of blocks in $I(q, 3)$ are given in Table 6.1.

For $q = 11$, apply Proposition 5.2 to a $\{3\}$-DGDD of type $5^1 1^6$, which exists by Lemma 3.9 and Theorem 3.14, to conclude that $\{45, 46, 47\} \subseteq \mathrm{Spec}(11, 3, 1)$. Membership of 41, 42, 43, and 44 in $\mathrm{Spec}(11, 3, 1)$ is given by Table 6.2.

For $q = 14$, apply Proposition 5.2 to a $\{3\}$-DGDD of type $4^3 2^1$, which exists by Lemma 3.9 and Theorem 3.14, to conclude that $[68, 74] \subseteq \mathrm{Spec}(14, 3, 1)$. Membership of 66 and 67 in $\mathrm{Spec}(14, 3, 1)$ is given by Table 6.2. ∎

Table 6.1
*Some perfect* $(3,1)_q$-DCCs *with small* $q$.

| $q$ | Code size | Codewords of a perfect $(3,1)_q$-DCC |
|---|---|---|
| 1 | 1 | (0, 0, 0) |
| 2 | 2 | (0, 0, 0) (1, 0, 1) |
| 3 | 3 | (0, 1, 0) (1, 2, 1) (2, 0, 2) |
| | 4 | (0, 0, 1) (0, 2, 2) (1, 2, 0) (2, 1, 1) |
| | 5 | (0, 0, 0) (0, 2, 1) (1, 1, 1) (1, 2, 0) (2, 2, 2) |
| 4 | 6 | (0, 0, 2) (0, 1, 3) (1, 2, 1) (2, 2, 0) (3, 1, 0) (3, 2, 3) |
| | 7 | (0, 0, 0) (0, 1, 2) (1, 3, 3) (2, 0, 3) (2, 1, 1) (3, 1, 0) (3, 2, 2) |
| | 8 | (0, 0, 0) (0, 3, 2) (1, 1, 1) (1, 2, 3) (2, 0, 1) (2, 2, 2) (3, 1, 0) (3, 3, 3) |
| 5 | 9 | (0, 3, 2) (0, 4, 0) (1, 2, 3) (1, 4, 4) (2, 1, 0) (2, 4, 2) (3, 0, 1) (3, 4, 3) (4, 1, 1) |
| | 10 | (0, 1, 0) (0, 4, 2) (1, 1, 2) (1, 3, 3) (2, 1, 4) (2, 3, 0) (3, 2, 2) (3, 4, 1) (4, 0, 3) (4, 4, 4) |
| | 11 | (0, 0, 0) (0, 1, 3) (0, 2, 4) (1, 1, 1) (1, 2, 0) (2, 2, 2) (2, 3, 1) (3, 3, 3) (3, 4, 2) (4, 1, 4) (4, 3, 0) |
| 6 | 12 | (0, 1, 0) (1, 5, 1) (2, 0, 2) (2, 1, 4) (3, 0, 4) (3, 1, 3) (3, 2, 5) (4, 0, 3) (4, 1, 2) (4, 5, 4) (5, 0, 5) (5, 2, 3) |
| | 13 | (0, 3, 1) (0, 4, 0) (1, 1, 3) (1, 2, 0) (1, 4, 5) (2, 3, 2) (2, 5, 4) (3, 0, 5) (4, 2, 1) (4, 3, 4) (5, 0, 2) (5, 3, 3) (5, 5, 1) |
| | 14 | (0, 1, 1) (0, 3, 2) (0, 4, 5) (1, 0, 0) (2, 1, 4) (2, 2, 2) (2, 3, 0) (3, 1, 5) (3, 4, 3) (4, 1, 2) (4, 4, 4) (5, 1, 3) (5, 2, 5) (5, 4, 0) |
| | 15 | (0, 2, 2) (0, 4, 1) (1, 1, 1) (1, 3, 4) (1, 5, 0) (2, 0, 3) (2, 5, 1) (3, 0, 5) (3, 1, 2) (3, 3, 3) (4, 0, 0) (4, 4, 2) (4, 5, 3) (5, 2, 4) (5, 5, 5) |
| | 16 | (0, 0, 0) (0, 4, 1) (0, 5, 3) (1, 1, 1) (1, 3, 5) (2, 1, 0) (2, 2, 2) (2, 4, 3) (3, 1, 2) (3, 3, 3) (3, 4, 0) (4, 2, 5) (4, 4, 4) (5, 0, 2) (5, 1, 4) (5, 5, 5) |
| 8 | 22 | (5, 1, 7) (5, 3, 6) (4, 0, 4) (3, 0, 1) (6, 4, 5) (1, 4, 6) (6, 0, 2) (6, 1, 3) (3, 4, 3) (2, 4, 2) (3, 2, 7) (6, 7, 6) (1, 2, 1) (7, 5, 4) (1, 5, 0) (7, 0, 0) (7, 2, 3) (4, 7, 1) (0, 3, 5) (5, 2, 5) (0, 7, 7) (2, 0, 6) |
| | 23 | (7, 7, 6) (2, 6, 1) (0, 3, 6) (7, 5, 1) (6, 5, 0) (1, 6, 7) (2, 5, 2) (6, 3, 2) (1, 0, 2) (0, 0, 5) (2, 4, 3) (5, 5, 7) (4, 5, 6) (1, 5, 3) (0, 7, 4) (3, 0, 1) (4, 7, 2) (1, 4, 1) (2, 7, 0) (3, 5, 4) (6, 6, 4) (4, 4, 0) (3, 7, 3) |
| | 24 | (7, 7, 3) (5, 4, 6) (3, 1, 5) (6, 2, 3) (5, 1, 1) (7, 4, 2) (3, 4, 0) (6, 7, 5) (0, 6, 0) (7, 6, 6) (4, 3, 7) (5, 0, 3) (2, 1, 6) (0, 1, 7) (2, 2, 2) (3, 3, 6) (1, 3, 2) (4, 5, 5) (2, 0, 4) (2, 5, 7) (7, 1, 0) (6, 4, 1) (0, 5, 2) (1, 4, 4) |
| | 25 | (0, 2, 5) (6, 6, 6) (3, 3, 2) (0, 1, 3) (3, 6, 0) (5, 1, 1) (1, 5, 5) (2, 2, 7) (7, 0, 6) (3, 7, 7) (1, 2, 6) (6, 7, 4) (5, 7, 2) (7, 5, 3) (4, 5, 6) (3, 1, 4) (4, 7, 1) (4, 2, 3) (6, 2, 1) (5, 4, 0) (0, 0, 0) (2, 0, 4) (6, 3, 5) (4, 4, 4) (1, 0, 7) |
| | 26 | (3, 1, 4) (2, 0, 7) (6, 2, 3) (4, 0, 5) (6, 6, 6) (2, 1, 5) (5, 7, 1) (2, 4, 6) (5, 4, 2) (1, 1, 1) (2, 2, 2) (0, 1, 2) (1, 0, 3) (4, 4, 4) (4, 3, 7) (7, 0, 4) (0, 0, 0) (3, 3, 5) (7, 7, 7) (7, 6, 5) (5, 6, 0) (7, 3, 2) (5, 5, 3) (1, 6, 7) (6, 4, 1) (3, 0, 6) |

Lemma 6.2. $\mathrm{Spec}(q,3,1) = I(q,3)$ *for all* $q \equiv 0$ *or* $1 \pmod 3$.

*Proof.* The case $q \le 6$ is settled by Lemma 6.1. If $q \equiv 0 \pmod 3$, $q \ge 9$, apply Proposition 5.2 to a $\{3\}$-DGDD of type $3^{q/3}$, obtained by directing a $(\{3\},2)$-GDD

TABLE 6.2
*Some perfect* $(3,1)_q$*-DCCs with* $q \in \{11, 14\}$.

| $q$ | Code size | Codewords of a perfect $(3,1)_q$-DCC |
|---|---|---|
| 11 | 41 | (5, 0, 8) (6, 7, 9) (5, 1, 5) (9, 7, 2) (8, 7, 10) (0, 1, 9) |
| | | (9, 3, 9) (1, 7, 8) (1, 2, 0) (4, 9, 8) (9, 10, 1) (2, 1, 3) |
| | | (8, 6, 1) (4, 10, 7) (4, 3, 6) (2, 7, 6) (6, 5, 6) (8, 2, 8) |
| | | (3, 10, 5) (3, 8, 4) (2, 2, 4) (6, 8, 0) (5, 3, 7) (7, 0, 7) |
| | | (0, 6, 4) (2, 5, 9) (7, 3, 1) (10, 8, 3) (6, 3, 2) (4, 4, 2) |
| | | (0, 5, 2) (5, 10, 4) (1, 6, 10) (3, 0, 3) (1, 4, 1) (9, 4, 0) |
| | | (7, 4, 5) (8, 9, 5) (10, 2, 10) (10, 9, 6) (0, 10, 0) |
| | 42 | (0, 9, 2) (9, 8, 3) (7, 9, 5) (3, 0, 3) (5, 4, 3) (2, 3, 4) |
| | | (6, 3, 7) (5, 1, 9) (3, 1, 5) (7, 8, 0) (5, 8, 10) (9, 0, 7) |
| | | (3, 8, 2) (10, 1, 3) (10, 10, 7) (3, 10, 9) (9, 4, 4) (4, 2, 5) |
| | | (9, 6, 10) (10, 5, 6) (8, 1, 8) (0, 1, 10) (8, 5, 7) (7, 4, 10) |
| | | (0, 6, 0) (6, 5, 2) (8, 6, 9) (1, 6, 1) (2, 6, 8) (6, 4, 6) |
| | | (1, 2, 7) (4, 7, 1) (4, 0, 8) (2, 2, 10) (10, 2, 0) (7, 7, 2) |
| | | (4, 9, 9) (5, 0, 5) (10, 8, 4) (2, 9, 1)(7, 3, 6) (1, 0, 4) |
| | 43 | (8, 7, 0) (10, 6, 3) (5, 10, 8) (8, 3, 3) (0, 4, 9) (5, 2, 1) |
| | | (6, 1, 1) (0, 0, 5) (4, 7, 2) (2, 9, 0) (5, 4, 6) (10, 1, 5) |
| | | (7, 5, 7) (10, 0, 7) (6, 8, 9) (10, 2, 4) (1, 7, 9) (9, 3, 7) |
| | | (9, 5, 9) (3, 0, 6) (0, 1, 8) (7, 3, 1) (2, 6, 6) (4, 1, 0) |
| | | (0, 2, 3) (3, 8, 2) (2, 8, 5) (7, 8, 6) (8, 8, 10) (0, 10, 10) |
| | | (9, 8, 1) (6, 2, 7) (6, 5, 0) (7, 4, 10) (9, 2, 10) (3, 10, 9) |
| | | (1, 4, 3) (5, 5, 3) (3, 4, 5) (9, 6, 4) (1, 6, 10) (1, 2, 2) |
| | | (4, 8, 4) |
| | 44 | (7, 9, 3) (10, 0, 9) (2, 8, 8) (6, 6, 1) (7, 8, 6) (6, 7, 4) |
| | | (7, 10, 2) (7, 1, 1) (0, 7, 7) (0, 3, 10) (6, 3, 0) (3, 1, 5) |
| | | (8, 2, 5) (4, 7, 5) (10, 4, 4) (5, 7, 0) (2, 10, 6) (9, 4, 0) |
| | | (5, 10, 10) (4, 8, 10) (2, 2, 3) (8, 3, 9) (0, 5, 8) (2, 1, 0) |
| | | (0, 4, 6) (4, 3, 2) (3, 3, 4) (9, 2, 7) (9, 8, 1) (1, 3, 6) |
| | | (2, 4, 9) (0, 1, 2) (5, 4, 1) (1, 9, 10) (1, 8, 4) (10, 5, 3) |
| | | (9, 5, 5) (6, 10, 8) (10, 1, 7) (3, 8, 7) (8, 0, 0) (6, 9, 9) |
| | | (6, 5, 2) (5, 9, 6) |
| 14 | 66 | (4, 7, 6) (8, 2, 1) (4, 12, 1) (2, 5, 9) (13, 5, 7) (0, 6, 11) |
| | | (7, 4, 11) (12, 0, 10) (6, 2, 6) (10, 12, 6) (1, 7, 8) (7, 13, 1) |
| | | (9, 8, 10) (11, 10, 13) (9, 5, 1) (5, 6, 4) (2, 0, 3) (10, 8, 7) |
| | | (5, 0, 5) (0, 9, 2) (1, 4, 10) (2, 12, 2) (8, 13, 11) (5, 11, 8) |
| | | (11, 6, 7) (13, 9, 12) (3, 1, 1) (2, 11, 4) (9, 13, 4) (2, 7, 10) |
| | | (3, 0, 12) (13, 10, 2) (12, 4, 13) (6, 0, 8) (1, 3, 3) (4, 9, 0) |
| | | (0, 1, 0) (8, 12, 8) (10, 1, 9) (11, 3, 11) (1, 11, 5) (6, 13, 3) |
| | | (11, 1, 2) (3, 7, 2) (9, 7, 3) (13, 0, 13) (5, 2, 13) (12, 7, 9) |
| | | (1, 13, 6) (10, 3, 10) (3, 13, 8) (3, 6, 9) (12, 11, 12) (10, 4, 5) |
| | | (6, 1, 12) (6, 5, 10) (8, 5, 3) (10, 11, 0) (8, 0, 4) (4, 3, 4) |
| | | (12, 3, 5) (7, 0, 7) (7, 5, 12) (8, 9, 6) (4, 2, 8) (9, 11, 9) |
| | 67 | (6, 5, 10) (8, 0, 8) (12, 5, 3) (13, 3, 3) (13, 10, 11) (3, 10, 10) |
| | | (6, 12, 6) (10, 2, 6) (6, 13, 9) (11, 3, 5) (9, 4, 10) (8, 11, 7) |
| | | (8, 13, 5) (12, 0, 2) (10, 4, 4) (3, 8, 2) (4, 8, 12) (9, 7, 12) |
| | | (2, 8, 9) (4, 9, 2) (13, 6, 7) (10, 13, 8) (9, 0, 13) (3, 12, 4) |
| | | (5, 4, 13) (5, 0, 6) (10, 5, 12) (0, 4, 3) (11, 6, 2) (2, 3, 1) |
| | | (11, 8, 1) (1, 10, 3) (9, 6, 11) (5, 9, 5) (8, 6, 3) (4, 1, 6) |
| | | (0, 1, 5) (12, 9, 1) (12, 8, 10) (7, 4, 11) (11, 0, 11) (11, 10, 9) |
| | | (5, 7, 8) (7, 3, 7) (12, 11, 12) (0, 7, 9) (1, 0, 12) (1, 13, 1) |
| | | (5, 1, 2) (6, 8, 4) (13, 4, 0)(3, 11, 13) (0, 10, 0) (13, 13, 2) |
| | | (2, 7, 0) (1, 11, 4) (10, 1, 7) (9, 3, 9) (2, 5, 11) (4, 7, 5) |
| | | (1, 9, 8) (7, 2, 10) (3, 6, 0) (12, 7, 13) (2, 2, 4) (7, 6, 1) |
| | | (2, 13, 12) |

of type $3^{q/3}$, which exists by Theorem 3.12, to conclude that

$$\begin{aligned}\mathrm{Spec}(q,3,1) &\supseteq \left\{\frac{q(q-3)}{3}\right\} + \sum_{i=1}^{q/3} \mathrm{Spec}(3,3,1)\\ &= \left[\frac{q^2}{3}, \frac{q^2+2q}{3}\right]\\ &= I(q,3).\end{aligned}$$

If $q \equiv 1 \pmod 3$, apply Proposition 5.2 to a $\{3\}$-DGDD of type $3^{(q-1)/3}1^1$, obtained by directing a $(\{3\},2)$-GDD of type $3^{(q-1)/3}1^1$, to conclude that

$$\begin{aligned}\mathrm{Spec}(q,3,1) &\supseteq \left\{\frac{(q-1)(q-2)}{3}+1\right\} + \sum_{i=1}^{(q-1)/3} \mathrm{Spec}(3,3,1)\\ &= \left[\frac{q^2+2}{3}, \frac{q^2+2q}{3}\right]\\ &= I(q,3).\end{aligned}$$

A $(\{3\},2)$-GDD of type $3^{(q-1)/3}1^1$ can be constructed by taking a near-resolvable $\mathrm{B}(q,3,2)$, whose existence is well established (see, for example, [16]), and considering one of its near-parallel class as the set of groups. □

Lemma 6.3. $\mathrm{Spec}(q,3,1) = I(q,3)$ *for all* $q \equiv 2 \pmod 3$.

*Proof.* The case $q \in \{8, 11, 14\}$ is settled by Lemma 6.1. If $q \equiv 2 \pmod 6$, $q \geq 20$, apply Proposition 5.2 to a $\{3\}$-DGDD of type $6^{(q-2)/6}2^1$, which exists by Lemma 3.9 and Theorem 3.14, to conclude that

$$\begin{aligned}\mathrm{Spec}(q,3,1) &\supseteq \left\{\frac{(q-2)(q-4)}{3}\right\} + \sum_{i=1}^{(q-2)/6} \mathrm{Spec}(6,3,1) + \mathrm{Spec}(2,3,1)\\ &= \left[\frac{q^2+2}{3}, \frac{q^2+2q-2}{3}\right]\\ &= I(q,3).\end{aligned}$$

If $q \equiv 5 \pmod 6$, $q \geq 17$, apply Proposition 5.2 to a $\{3\}$-DGDD of type $3^{(q-5)/3}5^1$, which exists by Lemma 3.9 and Theorem 3.14, to conclude that

$$\begin{aligned}\mathrm{Spec}(q,3,1) &\supseteq \left\{\frac{(q+2)(q-5)}{3}\right\} + \sum_{i=1}^{(q-5)/3} \mathrm{Spec}(3,3,1) + \mathrm{Spec}(5,3,1)\\ &= \left[\frac{q^2+2}{3}, \frac{q^2+2q-2}{3}\right]\\ &= I(q,3). \qquad \square\end{aligned}$$

Lemmas 6.2 and 6.3 combine to give the following.

Theorem 6.4. $\mathrm{Spec}(q,3,1) = I(q,3)$ *for all* $q$.

Table 7.1
*Some perfect $(4,2)_q$-DCCs with small $q$.*

| $q$ | Code size | Codewords of a perfect $(4,2)_q$-DCC |
|---|---|---|
| 1 | 1 | $(0,0,0,0)$ |
| 2 | 1 | $(0,1,1,0)$ |
| | 2 | Exists by Theorem 2.3 |
| 3 | 2 | $(0,2,1,0)$ $(1,1,2,2)$ |
| | 3 | Exists by Theorem 2.3 |
| 4 | 3 | Does not exist by Lemma 7.1 |
| | 4 | $(0,2,0,1)$ $(1,3,0,3)$ $(2,2,2,3)$ $(3,1,1,2)$ |
| | 5 | $(0,0,0,2)$ $(0,1,1,1)$ $(0,3,3,3)$ $(2,1,3,0)$ $(3,1,2,2)$ |
| | 6 | Exists by Theorem 2.3 |
| 5 | 5 | $(0,1,0,2)$ $(1,4,1,3)$ $(2,3,2,1)$ $(3,0,3,4)$ $(4,2,4,0)$ |
| | 6 | $(0,4,2,3)$ $(1,1,1,3)$ $(1,2,0,0)$ $(2,1,4,4)$ $(3,3,2,2)$ $(3,4,0,1)$ |
| | 7 | Exists by Theorem 2.3 |
| 6 | 6 | $(1,2,0,1)$ $(3,1,4,3)$ $(4,0,2,4)$ $(5,4,1,5)$ $(0,3,5,0)$ $(2,5,3,2)$ |
| | 7 | Does not exist by Lemma 7.1 |
| | 8 | $(4,2,5,5)$ $(5,2,0,3)$ $(0,2,4,1)$ $(1,3,0,5)$ $(3,1,2,2)$ $(4,0,0,0)$ $(3,4,4,3)$ $(5,1,1,4)$ |
| | 9 | Exists by Theorem 2.2 |
| | 10 | Exists by Theorem 2.3 |
| 7 | 9 | $(0,1,2,0)$ $(1,3,4,1)$ $(0,3,5,3)$ $(2,3,6,2)$ $(4,2,4,5)$ $(4,6,3,0)$ $(5,0,6,4)$ $(5,5,2,1)$ $(6,1,6,5)$ |
| | 10 | $(6,4,1,5)$ $(3,3,5,6)$ $(4,2,4,3)$ $(0,4,0,6)$ $(1,0,1,3)$ $(6,3,0,2)$ $(2,1,2,6)$ $(5,3,1,4)$ $(5,2,0,5)$ $(6,6,6,6)$ |
| | 11 | $(6,3,6,5)$ $(4,6,1,0)$ $(3,0,3,2)$ $(5,5,5,4)$ $(1,5,6,2)$ $(4,2,5,3)$ $(2,0,6,4)$ $(1,3,1,4)$ $(0,5,0,1)$ $(2,2,2,1)$ $(4,4,4,4)$ |
| | 12 | $(3,2,1,0)$ $(6,6,6,1)$ $(0,0,0,3)$ $(1,1,5,2)$ $(0,2,4,5)$ $(4,0,1,6)$ $(5,5,6,0)$ $(3,3,6,5)$ $(5,1,3,4)$ $(6,4,2,3)$ $(2,2,2,6)$ $(4,4,4,4)$ |
| | 13 | $(6,6,5,6)$ $(0,6,4,2)$ $(3,3,0,3)$ $(6,0,0,0)$ $(2,3,6,1)$ $(4,0,5,1)$ $(1,1,1,1)$ $(4,4,4,4)$ $(5,5,5,3)$ $(1,2,5,0)$ $(2,2,2,2)$ $(3,5,2,4)$ $(1,4,6,3)$ |
| | 14 | Exists by Theorem 2.3 |

## 7. The spectrum $\mathrm{Spec}(q,4,2)$.

**7.1. Nonexistence.** We begin with some nonexistence results.

Lemma 7.1. *The following deletion-correcting codes do not exist:*

1. *Perfect* $(4,2)_4$*-DCCs of size* $DL(4,4)=3$*; and*
2. *perfect* $(4,2)_6$*-DCCs of size* $DL(6,4)+1=7$.

*Proof.* The proof is established by exhaustive search. ∎

Next, we determine some elements of $\mathrm{Spec}(q,4,2)$ for small $q$, which are required in our recursive constructions later.

**7.2. Small spectrum members.**

Lemma 7.2. *For* $q \in \{1,2,3,5,7\}$*, we have* $\mathrm{Spec}(q,4,2)=I(q,4)$*, and*

1. $\mathrm{Spec}(4,4,2)=I(4,4)\setminus\{3\}$;
2. $\mathrm{Spec}(6,4,2)=I(6,4)\setminus\{7\}$.

*Proof.* For $q\in[7]$, the existence of perfect $(4,2)_q$-DCCs of the required sizes in $\mathrm{Spec}(q,4,2)$ are given in Table 7.1. ∎

Lemma 7.3. $DL(10,4)=18\in\mathrm{Spec}(10,4,2)$.

*Proof.* Take the alphabet $X$ to be $\{0, 1, \ldots, 9\}$. The blocks

$$\begin{array}{cccccc} (6,7,6,9) & (3,4,6,1) & (9,3,7,5) & (9,1,0,9) & (5,2,9,6) & (7,1,7,3) \\ (1,2,1,8) & (5,1,5,4) & (8,0,1,6) & (4,7,0,4) & (8,4,3,9) & (8,5,7,8) \\ (2,0,7,2) & (9,4,8,2) & (6,2,4,5) & (6,3,0,8) & (0,5,0,3) & (3,3,2,3) \end{array}$$

form a perfect $(4, 2)_{10}$-DCC of size 18. ∎

LEMMA 7.4. $DL(12, 4) = 24 \in \mathrm{Spec}(12, 4, 2)$.

*Proof.* Take the alphabet $X$ to be $\mathbb{Z}_{12}$. The orbits of the eight codewords

$$\begin{array}{cccc} (1,2,10,4) & (10,3,5,10) & (0,11,10,8) & (4,6,10,9) \\ (3,6,7,3) & (5,9,7,5) & (4,1,7,8) & (4,11,5,4) \end{array}$$

under the action of adding 4 modulo 12 form a perfect $(4, 2)_{12}$-DCC of size 24. ∎

LEMMA 7.5. $DL(12, 4) + 1 = 25 \in \mathrm{Spec}(12, 4, 2)$.

*Proof.* Take the alphabet $X$ to be $\mathbb{Z}_2 \times \{0, 1, 2, 3, 4, 5\}$. The orbits of the 12 codewords

$$\begin{array}{ccc} ((0,1),(0,1),(0,0),(1,1)) & ((0,2),(0,0),(0,1),(0,2)) & ((0,3),(1,0),(0,2),(0,3)) \\ ((0,4),(0,1),(1,2),(0,4)) & ((0,5),(0,2),(1,2),(0,5)) & ((0,0),(0,3),(0,4),(0,5)) \\ ((0,1),(0,3),(1,5),(1,3)) & ((0,2),(1,3),(1,1),(0,4)) & ((0,4),(0,0),(1,5),(1,4)) \\ ((0,4),(1,3),(0,2),(1,0)) & ((0,5),(1,1),(1,5),(0,0)) & ((0,5),(1,4),(1,3),(0,1)) \end{array}$$

under the action of $\mathbb{Z}_2 \times \{0, 1, 2, 3, 4, 5\}$, together with the additional codeword $((0, 0), (1, 0), (1, 0), (0, 0))$, form a perfect $(4, 2)_{12}$-DCC of size 25. ∎

LEMMA 7.6. $[26, 30] \subseteq \mathrm{Spec}(12, 4, 2)$.

*Proof.* There exists a $\{4\}$-DGDD of type $3^4$ by Lemma 3.9 and Theorem 3.15. Now apply Proposition 5.2 to obtain

$$\begin{aligned} \mathrm{Spec}(12, 4, 2) &\supseteq \{18\} + \sum_{i=1}^{4} \mathrm{Spec}(3, 4, 2) \\ &= [26, 30]. \end{aligned}$$ ∎

LEMMA 7.7. $\{30, 37, 38, 39\} \subseteq \mathrm{Spec}(13, 4, 2)$.

*Proof.* Take a perfect IDP$(13 : 1, 4)$ (which exists by Lemma 4.1) and fill in the hole of size one with a perfect $(4, 2)_1$-DCC of size one to obtain a $(4, 2)_{13}$-DCC of size 30.

Now, take a B$(13, 4, 1)$ (which exists by Theorem 3.1) and construct a $\{4\}$-GDD of type $1^9 4^1$ by considering a block as a group. It follows from Lemma 3.9 that there exists a $\{4\}$-DGDD of type $1^9 4^1$. Applying Proposition 5.2 then gives $37, 38, 39 \in \mathrm{Spec}(13, 4, 2)$. ∎

LEMMA 7.8. $DL(16, 4) = 44 \in \mathrm{Spec}(16, 4, 2)$.

*Proof.* Take the alphabet $X$ to be $\mathbb{Z}_8 \times \{0, 1\}$. The orbits of the five codewords

$$\begin{array}{ccc} ((0,0),(1,0),(1,1),(0,0)) & ((0,0),(4,1),(6,1),(7,1)) & ((0,1),(2,0),(7,1),(4,1)) \\ ((0,1),(5,0),(0,1),(0,0)) & ((0,1),(6,1),(4,0),(0,0)) & \end{array}$$

under the action of $\mathbb{Z}_8 \times \{0, 1\}$, together with the four additional codewords

$$\begin{array}{cc} ((0,0),(2,0),(4,0),(6,0)) & ((6,0),(4,0),(2,0),(0,0)) \\ ((1,0),(3,0),(5,0),(7,0)) & ((7,0),(5,0),(3,0),(1,0)) \end{array}$$

form a perfect $(4, 2)_{16}$-DCC of size 44. ∎

LEMMA 7.9. $[48, 56] \subseteq \mathrm{Spec}(16, 4, 2)$.

*Proof.* Apply Proposition 5.2 to a $\{4\}$-DGDD of type $4^4$, which exists by Theorem 3.19, to conclude that

$$\begin{aligned}\mathrm{Spec}(16,4,2) &\supseteq \{32\} + \sum_{i=1}^{4} \mathrm{Spec}(4,4,2)\\ &= [48,56]. \qquad \square\end{aligned}$$

LEMMA 7.10. $\mathrm{Spec}(28, 4, 2) = [133, 154] = I(28, 4)$.

*Proof.* Take the alphabet $X$ to be $\mathbb{Z}_{14} \times \{0, 1\}$. The orbits of the nine codewords

$$\begin{array}{cc}((0,0),(1,0),(0,0),(0,1)) & ((0,1),(2,0),(4,0),(0,1))\\ ((0,0),(3,0),(7,0),(12,0)) & ((0,1),(8,0),(12,1),(11,1))\\ ((0,1),(3,1),(5,1),(12,0)) & ((0,0),(2,1),(8,1),(8,0))\\ ((0,0),(5,1),(9,1),(6,0)) & ((0,0),(6,1),(1,1),(11,0))\\ ((0,0),(11,1),(7,1),(10,0)) & \end{array}$$

under the action of $\mathbb{Z}_{14} \times \{0, 1\}$, together with the seven additional codewords

$$\begin{array}{cc}((0,1),(7,1),(1,1),(8,1)) & ((2,1),(9,1),(3,1),(10,1))\\ ((4,1),(11,1),(5,1),(12,1)) & ((6,1),(13,1),(7,1),(0,1))\\ ((8,1),(1,1),(9,1),(2,1)) & ((10,1),(3,1),(11,1),(4,1))\\ ((12,1),(5,1),(13,1),(6,1)) & \end{array}$$

form a perfect $(4, 2)_{28}$-DCC of size 133.

To show that $[134, 154] \subseteq \mathrm{Spec}(28, 4, 2)$, apply Proposition 5.2 to a $\{4\}$-DGDD of type $7^4$, which exists by Theorem 3.19, to conclude that

$$\begin{aligned}\mathrm{Spec}(28,4,2) &\supseteq \{98\} + \sum_{i=1}^{4} \mathrm{Spec}(7,4,2)\\ &= [134,154],\end{aligned}$$

since $\mathrm{Spec}(7, 4, 2) = [9, 14]$ and $\sum_{i=1}^{4}\{9, 10, 11, 12, 13, 14\} = [36, 56]$. $\square$

### 7.3. The case $q \equiv 0 \pmod 3$.

LEMMA 7.11. *$DL(q, 4) \in \mathrm{Spec}(q, 4, 2)$ for all $q \equiv 0 \pmod 6$, $q \geq 24$.*

*Proof.* For $q \geq 24$, apply Proposition 5.2 to a $\{4\}$-DGDD of type $6^{q/6}$, which exists by Theorem 3.19. $\square$

LEMMA 7.12. *$[DL(q, 4) + 1, DU(q, 4) - (\lfloor \frac{q}{12} \rfloor + 1)] \subseteq \mathrm{Spec}(q, 4, 2)$ for all $q \equiv 0 \pmod 6$, $q \geq 30$.*

*Proof.* By Theorem 3.18, there exist both a $\{4\}$-GDD of type $6^{(q-6)/6}3^1$ and a $\{4\}$-GDD of type $6^{(q-6)/6}6^1 3^1$. Apply Lemma 3.10 to obtain a $\{4\}$-DGDD of type $6^{(q-6)/6}3^2$. Now apply Proposition 5.2 to obtain

$$\begin{aligned}\mathrm{Spec}(q,4,2) &\supseteq \left\{\frac{q(q-6)}{6}\right\} + \sum_{i=1}^{2}\mathrm{Spec}(3,4,2) + \sum_{i=1}^{(q-6)/6}\mathrm{Spec}(6,4,2)\\ &= \left[\frac{q^2}{6}+1, \frac{q^2+4q-6}{6}\right]\\ &= \left[DL(q,4)+1, DU(q,4) - \left(\left\lfloor\frac{q}{12}\right\rfloor + 1\right)\right]. \qquad \square\end{aligned}$$

LEMMA 7.13. *$\left[DL(q,4), DU(q,4) - \left\lfloor \frac{q+3}{12} \right\rfloor\right] \subseteq \mathrm{Spec}(q,4,2)$ for all $q \equiv 3 \pmod 6$, $q \geq 27$.*

*Proof.* For $q \geq 27$, there exists a $\{4\}$-DGDD of type $6^{(q-3)/6}3^1$ by Lemma 3.9 and Theorem 3.18. Now apply Proposition 5.2 to obtain

$$\begin{aligned}\mathrm{Spec}(q,4,2) &\supseteq \left\{\frac{(q-3)^2}{6}\right\} + \mathrm{Spec}(3,4,2) + \sum_{i=1}^{(q-3)/6} \mathrm{Spec}(6,4,2) \\ &= \left[\frac{q^2+3}{6}, \frac{q^2+4q-3}{6}\right] \\ &= \left[DL(q,4), DU(q,4) - \left\lfloor \frac{q+3}{12} \right\rfloor\right]. \qquad \square\end{aligned}$$

LEMMA 7.14. *There exists a perfect* DP$(12t, 4)$ *of size belonging to* $[DU(12t,4) - 6t, DU(12t,4)]$ *for all* $t \geq 1$.

*Proof.* An RDP$(12t, 4)$ is known [3]. The leave graph $\Gamma$ of this RDP$(12t, 4)$ is 1-regular. For each $j \in [0, 6t]$, adding $j$ blocks $(a, a, b, b)$ for $j$ arcs $(a, b) \in E(\Gamma)$ and $12t - 2j$ blocks $(c, c, c, d)$ and $(d, d, d, d)$ for the remaining arcs $(c, d) \in E(\Gamma)$ to this RDP$(12t, 4)$ gives a perfect DP$(12t, 4)$ having $DU(12t, 4) - j$ blocks. $\square$

LEMMA 7.15. *There exists a perfect* DP$(12t+3, 4)$ *of size belonging to* $[DU(12t+3,4) - 6t, DU(12t+3,4)]$ *for all* $t \geq 1$.

*Proof.* Let $(\{0, 1, \ldots, 12t+1\}, \mathcal{G}, \mathcal{A})$ be a $\{4\}$-GDD of type $2^{6t+1}$, which exists by Theorem 3.15, with $\mathcal{G} = \{\{2i, 2i+1\} : 0 \leq i \leq 6t\}$, and let $(\{0, 1, \ldots, 12t+3\}, \mathcal{B})$ be a B$(12t+4, 4, 1)$, which exists by Theorem 3.1, such that $[12t, 12t+3] \in \mathcal{B}$. Let $\mathcal{C} = \mathcal{B} \setminus \{[12t, 12t+3]\}$, and let $\mathcal{C}(x) = \{A \in \mathcal{C} : x \in A\}$. Then $\mathcal{C}(12t+2)$ and $\mathcal{C}(12t+3)$ are disjoint. Define

$$\begin{aligned}\mathcal{A}_1 &= \{A_\uparrow : A \in \mathcal{A}\},\\ \mathcal{A}_2 &= \{A_\downarrow : A \in \mathcal{C} \setminus (\mathcal{C}(12t+2) \cup \mathcal{C}(12t+3))\},\\ \mathcal{A}_3 &= \{(12t+2, a, b, c) : a > b > c \text{ and } \{a, b, c, 12t+2\} \in \mathcal{C}(12t+2)\},\\ \mathcal{A}_4 &= \{(a, b, c, 12t+2) : a > b > c \text{ and } \{a, b, c, 12t+3\} \in \mathcal{C}(12t+3)\}.\end{aligned}$$

Then $(\{0, 1, \ldots, 12t+2\}, \cup_{i=1}^4 \mathcal{A}_i)$ is an RDP$(12t+3, 4)$ of size $|\mathcal{A}| + (|\mathcal{B}| - 1) = 2t(6t+1) + (4t+1)(3t+1) - 1 = 3t(8t+3) = RDU(12t+3, 4)$.

For each $j \in [0, 6t]$, to obtain a perfect DP$(12t+3, 4)$ of size $DU(12t+3, 4) - j$ with this RDP$(12t+3, 4)$ as a subsystem, we add to it the following $12t+3-j$ blocks:

$$\begin{aligned}&(12t, 12t, 12t+1, 12t+2),\\ &(12t+2, 12t+1, 12t+1, 12t),\\ &(12t+2, 12t+2, 12t+2, 12t+2),\\ &(2i, 2i, 2i+1, 2i+1) \text{ for } 0 \leq i \leq j-1,\\ &(2i, 2i, 2i, 2i+1) \text{ for } j \leq i \leq 6t-1, \text{ and}\\ &(2i+1, 2i+1, 2i+1, 2i+1) \text{ for } j \leq i \leq 6t-1. \qquad \square\end{aligned}$$

LEMMA 7.16. *There exists a perfect* DP$(12t+6, 4)$ *of size belonging to* $[DU(12t+6,4) - 6t, DU(12t+6,4)]$ *for all* $t \geq 1$.

*Proof.* Let $(\{0, 1, \ldots, 12t+7\}, \mathcal{G}, \mathcal{A})$ be a $\{4\}$-GDD of type $2^{6t+4}$, which exists by Theorem 3.15. Without loss of generality, assume that $[12t+4, 12t+7] \in \mathcal{A}$, and let $(\{0, 1, \ldots, 12t+3\}, \mathcal{B})$ be a B$(12t+4, 4, 1)$, which exists by Theorem 3.1. Let

$\mathcal{C} = \mathcal{A} \setminus \{[12t+4, 12t+7]\}$, and let $\mathcal{C}(x) = \{A \in \mathcal{C} : x \in A\}$. Then $\mathcal{C}(12t+4)$, $\mathcal{C}(12t+5)$, $\mathcal{C}(12t+6)$, and $\mathcal{C}(12t+7)$ are pairwise disjoint. Define

$$
\begin{aligned}
\mathcal{A}_1 &= \{A_\uparrow : A \cap [12t+4, 12t+7] = \emptyset \text{ and } A \in \mathcal{C}\},\\
\mathcal{A}_2 &= \{(12t+4, a, b, c) : a < b < c \text{ and } \{a, b, c, 12t+4\} \in \mathcal{C}(12t+4)\},\\
\mathcal{A}_3 &= \{(12t+5, a, b, c) : a < b < c \text{ and } \{a, b, c, 12t+5\} \in \mathcal{C}(12t+5)\},\\
\mathcal{A}_4 &= \{(a, b, c, 12t+5) : a < b < c \text{ and } \{a, b, c, 12t+6\} \in \mathcal{C}(12t+6)\},\\
\mathcal{A}_5 &= \{(a, b, c, 12t+4) : a < b < c \text{ and } \{a, b, c, 12t+7\} \in \mathcal{C}(12t+7)\},\\
\mathcal{A}_6 &= \{B_\downarrow : B \in \mathcal{B}\}.
\end{aligned}
$$

Then $(\{0, 1, \ldots, 12t+5\}, \cup_{i=1}^{6} \mathcal{A}_i)$ is an $\mathrm{RDP}(12t+6, 4)$ of size $(|\mathcal{A}| - 1) + |\mathcal{B}| = 2(3t+2)(2t+1) - 1 + (4t+1)(3t+1) = 24t^2 + 21t + 4 = RDU(12t+6, 4)$.

For each $j \in [0, 6t]$, to derive a perfect $\mathrm{DP}(12t+6, 4)$ of size $DU(12t+6, 4) - j$ with this $\mathrm{RDP}(12t+6, 4)$ as a subsystem, we add $12t+6-j$ blocks to this $\mathrm{RDP}(12t+6, 4)$ as follows. Partition the set $\{\{a, b\} \in \mathcal{G}, a < b < 12t+4\}$ into two disjoint parts $\mathcal{G}_0$ and $\mathcal{G}_1$ such that $|\mathcal{G}_0| = j$. Hence, $|\mathcal{G}_1| = 6t - j$. Let

$$
\begin{aligned}
\mathcal{B}_0 &= \{(a, a, b, b) : \{a, b\} \in \mathcal{G}_0\},\\
\mathcal{B}_1 &= \{(a, a, a, a), (a, b, b, b) : \{a, b\} \in \mathcal{G}_1\},\\
\mathcal{B}_2 &= \{(a, a, a, 12t+4) : \{a, 12t+4\} \in \mathcal{G}\},\\
\mathcal{B}_3 &= \{(a, a, a, 12t+5) : \{a, 12t+5\} \in \mathcal{G}\},\\
\mathcal{B}_4 &= \{(a, a, a, 12t+5) : \{a, 12t+6\} \in \mathcal{G}\},\\
\mathcal{B}_5 &= \{(a, a, a, 12t+4) : \{a, 12t+7\} \in \mathcal{G}\}, \text{ and}\\
\mathcal{B}_6 &= \{(a, a, a, b), (b, b, b, a) : \{a, b\} = \{12t+4, 12t+5\}\}.
\end{aligned}
$$

Then $(\{0, 1, \ldots, 12t+5\}, \mathcal{B}_0 \cup_{i=1}^{6} (\mathcal{A}_i \cup \mathcal{B}_i))$ is a perfect $\mathrm{DP}(12t+6, 4)$ of size $DU(12t+6, 4) - j$. ∎

LEMMA 7.17. *There exists a perfect* $\mathrm{DP}(12t+9, 4)$ *of size belonging to* $[DU(12t+9, 4) - (6t+2), DU(12t+9, 4)]$ *for all* $t \geq 1$.

*Proof.* Let $(\{0, 1, \ldots, 12t+7\}, \mathcal{G}, \mathcal{A})$ be a $\{4\}$-GDD of type $2^{6t+4}$ with $\{12t+2i, 12t+2i+1\} \in \mathcal{G}$ for $i \in [3]$, which exists by Theorem 3.15, and let $(\{0, 1, \ldots, 12t+9\}, \mathcal{B})$ be a $\mathrm{PBD}(12t+10, \{4, 7^\star\})$ with $[12t+3, 12t+9]$ as the block of size seven, which exists by Theorem 3.8. Let $\mathcal{C} = \mathcal{B} \setminus \{[12t+3, 12t+9]\}$, and let $\mathcal{C}(x) = \{A \in \mathcal{C} : x \in A\}$. Define

$$
\begin{aligned}
\mathcal{A}_1 &= \{A_\uparrow : A \in \mathcal{A}\},\\
\mathcal{A}_2 &= \{(12t+7, 12t+6, 12t+8, 12t+5), (12t+8, 12t+6, 12t+7, 12t+4),\\
&\qquad (12t+5, 12t+4, 12t+3, 12t+8)\},\\
\mathcal{A}_3 &= \{A_\downarrow : A \in \mathcal{C} \setminus \mathcal{C}(12t+9)\},\\
\mathcal{A}_4 &= \{(a, b, c, 12t+8) : a > b > c \text{ and } \{a, b, c, 12t+9\} \in \mathcal{C}(12t+9)\}.
\end{aligned}
$$

Then $(\{0, 1, \ldots, 12t+8\}, \cup_{i=1}^{4} \mathcal{A}_i)$ is an $\mathrm{RDP}(12t+9, 4)$ of size $|\mathcal{A}| + (|\mathcal{B}| + 2) = 2(3t+2)(2t+1) + (4t+1)(3t+4) + 3 = RDU(12t+9, 4)$.

For each $j \in [0, 6t+2]$, to derive a perfect $\mathrm{DP}(12t+9, 4)$ of size $DU(12t+9, 4) - j$ with this $\mathrm{RDP}(12t+9, 4)$ as a subsystem, we add $12t+9-j$ blocks to this $\mathrm{RDP}(12t+9, 4)$ as follows. Partition the set $\{\{a, b\} \in \mathcal{G}, a < b < 12t+4\}$ into two disjoint parts

$\mathcal{G}_0$ and $\mathcal{G}_1$ such that $|\mathcal{G}_0| = j$. Hence, $|\mathcal{G}_1| = 6t + 2 - j$. Let

$$\begin{aligned}
\mathcal{B}_0 &= \{(a,a,b,b) : \{a,b\} \in \mathcal{G}_0\},\\
\mathcal{B}_1 &= \{(a,a,a,b),(b,b,b,b) : \{a,b\} \in \mathcal{G}_1\},\\
\mathcal{B}_2 &= \{(12t+i, 12t+i, 12t+i, 12t+3) : i \in \{6,7,8\}\},\\
\mathcal{B}_3 &= \{(12t+4, 12t+4, 12t+4, 12t+5)\}, \text{ and}\\
\mathcal{B}_4 &= \{(12t+5, 12t+5, 12t+5, 12t+4)\}.
\end{aligned}$$

Then $(\{0,1,\ldots,12t+8\}, \mathcal{B}_0 \cup_{i=1}^{4} (\mathcal{A}_i \cup \mathcal{B}_i))$ is a perfect DP$(12t+9,4)$ of size $DU(12t+9,4) - j$. ∎

We summarize these results as follows.

THEOREM 7.18. $\text{Spec}(q,4,2) = I(q,4)$ *for all* $q \equiv 0 \pmod 3$, *except when* $q = 6$, *and except possibly when* $q \in \{9,15,18,21,24\}$.

### 7.4. The case $q \equiv 1 \pmod 3$.

LEMMA 7.19. $\text{Spec}(q,4,2) = I(q,4)$ *for all* $q \equiv 1 \pmod{12}$, $q \geq 49$.

*Proof.* Take a B$(13,4,1)$ (which exists by Theorem 3.1) and construct a $\{4\}$-GDD of type $1^9 4^1$ by considering a block as a group. It follows from Lemma 3.9 that there exists a $\{4\}$-DGDD of type $1^9 4^1$. Applying Proposition 5.2 then gives an incomplete code with $x$ codewords, where $x \in \{36,37,38\}$, each with a hole of size one. We also have a perfect IDP$(13:1,4)$ (which exists by Lemma 4.1).

We first deal with $q \geq 61$. By Theorem 3.17, there exist both a $\{4\}$-GDD of type $12^{\frac{q-13}{12}} 6^1$ and a $\{4\}$-GDD of type $12^{\frac{q-13}{12}} 12^1 6^1$. Apply Lemma 3.10 to obtain a $\{4\}$-DGDD of type $12^{\frac{q-13}{12}} 6^2$. Add one point and apply Proposition 5.3. For every group of size 12, we put a copy of an incomplete code on 13 points with $x$ codewords, where $x \in \{29,36,37,38\}$, each having a hole of size one. For one of the group of size six, we put a copy of an incomplete code on seven points with $x$ codewords, where $x \in \{9,10,11,12,13\}$, each having a hole of size one. For the last group of size six, we put a $(4,2)_7$-DCC of size $x$, where $x \in [9,14]$. This gives the entire spectrum from $DL(q,4)$ to $DU(q,4)$.

For $q = 49$, take a $\{4\}$-DGDD of type $12^4$ and add one point, fill in the groups of size 12 with a $(4,2)_{13}$-DCC having a hole of size one and 29 codewords. Add one codeword for the hole of size one. This gives a $(4,2)_{49}$-DCC with $DL(49,4)$ codewords. Now, take a $\{4\}$-DGDD of type $7^7$ (which exists by Theorem 3.19). Fill in the groups with a $(4,2)_7$-DCC having $9,10,11,12,13$, or $14$ codewords; this gives the remaining cases. ∎

LEMMA 7.20. $\text{Spec}(q,4,2) = I(q,4)$ *for all* $q \equiv 4 \pmod{12}$, $q \geq 64$.

*Proof.* For the sizes belonging to $[DL(q,4)+1, DL(q,4)+3]$, take a $\{4\}$-DGDD of type $12^{\frac{q-4}{12}} 3^1$ (which exists by Theorem 3.17 and Lemma 3.9) and add one point. For every group of size 12, we put a copy of a perfect IDP$(13:1,4)$ with 29 blocks. For the last group, we put a $(4,2)_4$-DCC of size $x$, where $4 \leq x \leq 6$.

For the sizes belonging to $[DL(q,4)+4, DU(q,4)] \cup \{DL(q,4)\}$, take a $\{4\}$-DGDD of type $12^{\frac{q-16}{12}} 15^1$ (which exists by Theorem 3.17 and Lemma 3.9) and add one point. For every group of size 12, we put a copy of an incomplete code on 13 points with $x$ codewords, where $x \in \{29,36,37,38\}$, each having a hole of size one. For the last group, we put a $(4,2)_{16}$-DCC of size $x$, where $x \in \{44,48,49,50,51,52,53,54,55,56\}$. ∎

LEMMA 7.21. $\text{Spec}(q,4,2) = I(q,4)$ *for all* $q \equiv 7 \pmod{12}$, $q \geq 55$.

*Proof.* For the sizes belonging to $[DL(q,4)+1, DL(q,4)+5]$, take a $\{4\}$-DGDD of type $12^{\frac{q-7}{12}}6^1$ (which exists by Theorem 3.17 and Lemma 3.9) and add one point. For every group of size 12, we put a copy of a perfect IDP$(13:1,4)$ with 29 blocks. For the last group, we put a $(4,2)_7$-DCC of size $x$, where $9 \le x \le 14$.

Take a $\{4\}$-DGDD of type $6^5$ (which exists by Theorem 3.19), add one point, and apply Proposition 5.3. For four groups of size six, we put a copy of an incomplete code on seven points with $x$ codewords, where $x \in \{9,10,11,12,13\}$, each having a hole of size one. For the last group, we put a $(4,2)_7$-DCC of size $x$, where $x \in \{9,10,11,12,13,14\}$. This gives $[DL(31,4)+2, DU(31,4)] = [165,186] \subseteq \text{Spec}(31,4,2)$.

When $q \ge 103$, for the sizes belonging to $[DL(q,4)+6, DU(q,4)]$, take a $\{4\}$-DGDD of type $12^{\frac{q-31}{12}}30^1$ (which exists by Theorem 3.17 and Lemma 3.9) and add one point. For every group of size 12, we put a copy of an incomplete code on 13 points with $x$ codewords, where $x \in \{29,36,37,38\}$, each having a hole of size one. For the last group, we put a $(4,2)_{31}$-DCC of size $x$, where $x \in [165,186]$.

When $q \in \{55,67,79,91\}$, for the sizes belonging to $[DL(q,4)+6, DU(q,4)]$, we take a $\{4\}$-DGDD of type $6^9, 6^{11}, 6^{13}$, or $6^{13}12^1$ (which exists by Theorem 3.18 and Lemma 3.9), respectively, to obtain the desired codes. Here, we fill in one group of size six with a code on seven points and other groups with incomplete codes. ∎

LEMMA 7.22. *$\text{Spec}(q,4,2) = I(q,4)$ for all $q \equiv 10 \pmod{12}$, $q \ge 58$.*

*Proof.* Take a $\{4\}$-DGDD of type $12^{\frac{q-10}{12}}9^1$ (which exists by Theorem 3.17 and Lemma 3.9) and add one point. For every group of size 12, we put a copy of a perfect IDP$(13:1,4)$ with 29 blocks. For the last group, we put a $(4,2)_{10}$-DCC of size 18. This gives a $(4,2)_q$-DCC with $DL(q,4)$ codewords.

By Theorem 3.17, there exist both a $\{4\}$-GDD of type $12^{\frac{q-10}{12}}3^1$ and a $\{4\}$-GDD of type $12^{\frac{q-10}{12}}12^1 3^1$. Apply Lemma 3.10 to obtain a $\{4\}$-DGDD of type $12^{\frac{q-10}{12}}6^1 3^1$. Add one point and fill in the holes. For every group of size 12, we put a copy of an incomplete code on 13 points with $x$ codewords, where $x \in \{29,36,37,38\}$, each having a hole of size one. For the group of size six, we put a copy of an incomplete code on seven points with $x$ codewords, where $x \in \{9,10,11,12,13\}$, each having a hole of size one. For the last group of size three, we put a $(4,2)_4$-DCC of size $x$, where $x \in \{4,5,6\}$. This gives the entire spectrum from $DL(q,4)+1$ to $DU(q,4)$. ∎

We summarize these results as follows.

THEOREM 7.23. *$\text{Spec}(q,4,2) = I(q,4)$ for all $q \equiv 1 \pmod 3$, except when $q = 4$, and except possibly when $q \in \{10,13,16,19,22,25,31,34,37,40,43,46,52\}$.*

### 7.5. The case $q \equiv 2 \pmod 3$.

LEMMA 7.24. *$\text{Spec}(q,4,2) = I(q,4)$ for all $q \equiv 2 \pmod 6$.*

*Proof.* The case $q = 2$ is settled by Lemma 7.2. For $q \ge 8$, apply Proposition 5.2 to a $\{4\}$-DGDD of type $2^{q/2}$, which exists by Theorem 3.19, to conclude that

$$\begin{aligned}\text{Spec}(q,4,2) &\supseteq \left\{\frac{q(q-2)}{6}\right\} + \sum_{i=1}^{q/2} \text{Spec}(2,4,2)\\ &= \left[\frac{q(q+1)}{6}, \frac{q(q+4)}{6}\right]\\ &= I(q,4). \quad \square\end{aligned}$$

To deal with $q \equiv 5 \pmod 6$, we need a class of $\{4\}$-DGDDs that we establish below.

LEMMA 7.25. *A* $\{4\}$-DGDD *of type* $2^t5^1$ *exists if and only if* $t \equiv 0 \pmod 3$, $t \geq 6$.

*Proof.* We first establish necessity of the condition $t \equiv 0 \pmod 3$, $t \geq 6$. Suppose $x$ is a point contained in a group of size two. Then $x$ must appear with each of $2t+3$ points twice. Hence $2(2t+3) \equiv 0 \pmod 3$. This gives $t \equiv 0 \pmod 3$. Any $\{4\}$-DGDD must contain at least four groups, so a $\{4\}$-DGDD of type $5^1$ cannot exist. The nonexistence of a $\{4\}$-DGDD of type $2^35^1$ is easily established by computation. Hence $t \geq 6$.

To construct a $\{4\}$-DGDD of type $2^65^1$, take a resolvable $(\{3\}, 2)$-GDD $(X, \mathcal{G}, \mathcal{A})$ of type $2^6$, which exists by Theorem 3.13, and let $\{\mathcal{A}_1, \ldots, \mathcal{A}_{10}\}$ be a partition of $\mathcal{A}$ into parallel classes. This GDD is directable by Theorem 3.20 and hence underlies a $\{3\}$-DGDD $(X, \mathcal{G}, \mathcal{B})$ of type $2^6$. Now let $\infty_i \notin X$, $i \in [5]$, and form $\mathcal{C}$ as follows:

$$\begin{aligned} \mathcal{C} = &\{(a, b, c, \infty_i) : (a, b, c) \in \mathcal{B} \text{ and } \{a, b, c\} \in \mathcal{A}_i\} \\ &\cup \{(\infty_i, a, b, c) : (a, b, c) \in \mathcal{B} \text{ and } \{a, b, c\} \in \mathcal{A}_{i+5}\}. \end{aligned}$$

Then $(X, \mathcal{G} \cup \{\{\infty_1, \ldots, \infty_5\}\}, \mathcal{C})$ is a $\{4\}$-DGDD of type $2^65^1$.

For $t \geq 9$, a $\{4\}$-DGDD of type $2^t5^1$ exists by Lemma 3.9 and Theorem 3.16. ∎

LEMMA 7.26. $\mathrm{Spec}(q, 4, 2) = I(q, 4)$ *for all* $q \equiv 5 \pmod 6$, $q \neq 11$.

*Proof.* The case $q = 5$ is settled by Lemma 7.2. For $q \geq 17$, apply Proposition 5.2 to a $\{4\}$-DGDD of type $2^{(q-5)/2}5^1$, which exists by Lemma 7.25, to conclude that

$$\begin{aligned} \mathrm{Spec}(q, 4, 2) &\supseteq \left\{ \frac{(q+3)(q-5)}{6} \right\} + \sum_{i=1}^{(q-5)/2} \mathrm{Spec}(2, 4, 2) + \mathrm{Spec}(5, 4, 2) \\ &= \left[ \frac{q(q+1)}{6}, \frac{q^2+4q-3}{6} \right] \\ &= I(q, 4). \qquad \square \end{aligned}$$

We summarize these results as follows.

THEOREM 7.27. $\mathrm{Spec}(q, 4, 2) = I(q, 4)$ *for all* $q \equiv 2 \pmod 3$, *except possibly when* $q = 11$.

**7.6. Summary.**

THEOREM 7.28. $\mathrm{Spec}(q, 4, 2) = I(q, 4)$ *for all positive integers* $q$, *except when* $q \in \{4, 6\}$, *and except possibly when* $q \in \{9, 10, 11, 13, 15, 16, 18, 19, 21, 22, 24, 25, 31, 34, 37, 40, 43, 46, 52\}$.

**8. Conclusion.** Much work has been done on the existence of a perfect $(n, n-2)_q$-DCC. However, very little is known on the general problem of determining the spectrum of possible sizes for a perfect $(n, n-2)_q$-DCC. In this paper, we determine completely the spectrum of possible sizes for perfect $q$-ary 1-deletion-correcting codes of length three for all $q$, and perfect $q$-ary 2-deletion-correcting codes of length four for all but 19 values of $q$. A complete solution to these undetermined codes appears difficult.

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